4.1. Preliminary Theory – Linear Equations

1. Initial-Value and Boundary-Value Problems.

IVP: (1)

Solve: $$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to: $$y(x_0)=y_0,\ y'(x_0)=y_1,\ \ldots,\ y^{(n-1)}(x_0)=y_{n-1}$$

Theorem: (Existence of a Unique Solution)

Let $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x)\neq 0$ for every $x\in I$. If $x=x_0$ is any point in this interval, then a solution $y(x)$ of the IVP (1) exists on the interval and is unique.

Example 1. The IVP

$$3y'''+5y''-y'+7y=0$$
$$y(1)=0,\ y'(1)=0,\ y''(1)=0$$

has the trivial solution $y=0$. By the theorem this linear equation has only one solution $y=0$ for any interval containing $x=1$.

Example 2. It is easy to verify that

$$y(x) = 3e^{2x} + e^{-2x} - 3x$$ is a solution of the IVP

$$y'' - 4y = 12x \ , \ y(0) = 4, \ y'(0) = 1.$$

The eq is linear and $g(x) = 12x$ are continuous and $a_2(x) = 1 \neq 0$ on any interval I containing $x = 0$. Thus the eq has the above unique solution on I.

BOUNDARY-VALUE PROBLEM.

Solve: $$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Subject to: $$y(a) = y_0 , \ y(b) = y_1$$

is called a **boundary-value problem (BVP)**, the constraints $y(a) = y_0$ and $y(b) = y_1$ are called boundary condition.

The boundary condition of a second-order DE can be

$$(y'(a) = y_0 , \quad y(b) = y_1),$$

$$(y(a) = y_0 , \quad y'(b) = y_1)$$

or

$$(y'(a) = y_0 , \quad y'(b) = y_1).$$

In general the BC may be

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$

$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

Example. The eq: $y'' + 16y = 0$ has a family of solutions

$$y = c_1 \cos 4x + c_2 \sin 4x \qquad (2)$$

a) For example, we want to find the solution that further satisfies the boundary condition

$$y(0) = 0 \text{ and } y(\pi/2) = 0.$$

Let $x=0$: $c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

$\Rightarrow$

$$y = c_2 \sin 4x$$

Let $x = \pi/2$:

$$c_2 \sin 2\pi = 0 \Rightarrow c_2 \text{ can be any number}$$

$\Rightarrow$ The BVP:

$$y'' + 16y = 0$$
$$y(0) = 0, \; y(\pi/2) = 0$$

has infinitely many solutions.

$$x = c_2 \sin 4x.$$

that pass through the two points $(0,0)$ and $(\pi/2, 0)$.

(b) If we change the BC to $y(0) = 0$, $y(\frac{\pi}{8}) = 0$, we get $c_1 = 0$ from $y(0) = 0$.

for $y(\pi/8) = 0$ we get $c_2 \sin(\pi/2) = c_2 \cdot 1 = 0$

$\Rightarrow y = 0$ is the only solution.

(c)

If we consider the BC: $x(0) = 0$, $x(\frac{\pi}{2}) = 1$

then we have no solution.

2. HOMOGENEOUS EQUATIONS

The **homogeneous** equation has form

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x) = 0, \quad (6)$$

and the equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x) = g(x) \quad (7)$$

, for $g(x) \neq 0$, is said to be **nonhomogeneous**.

DIFFERENTIAL OPERATIONS

The differentiation $\frac{dy}{dx}$ is usually denoted by Dy, and D is called a **differential operator**.

For example:

$$D(\cos 4x) = -4\sin 4x$$
$$D(x^3 + 2x) = 3x^2 + 2.$$

$$\frac{d^2y}{dx^2} = D^2 y, \quad \frac{d^n y}{dx^n} = D^n y.$$

The polynomial operator has form

(8) $$L = a_n(x) D^n + a_{n-1}(x) D^{n-1} + \cdots + a_1(x) D + a_0(x)$$

<u>linear Property</u>:

$$L(\alpha f(x) + \beta g(x)) = \alpha L(f(x)) + \beta L(g(x))$$

The DE can be written in terms of D:

$$y''' + 6y'' - 7y' + 2y = 5x - 3$$

$$\Leftrightarrow \quad D^3 y + 6D^2 y - 7Dy + 2y = 5x - 3.$$

The linear eq can be written as
$Ly = g(x)$ as L is defined as in (8).

SUPERPOSITION PRINCIPLE.

<u>Theorem</u>. Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous n-th order equation (6) on an interval I. Then the linear combination

$$y = c_1 y_1(x) + \cdots + c_k y_k(x),$$

where c_i are arbitrary constants, is also a solution on the interval.

<u>Proof</u>:

Let L be the differential operation

$$L = a_n(x) D^n + \cdots + a_1(x) D + a_0(x)$$

Assume that $y_1, y_2, \ldots, y_k$ are solutions of the equation $L(y) = 0$.

If we define
$$y(x) = c_1 y_1(x) + \cdots + c_k y_k(x),$$
then $L(y) = L(c_1 y_1 + \cdots + c_k y_k)$

$$\begin{aligned} &= c_1 L(y_1) + \cdots + c_k L(y_k) \\ &= c_1\, 0 + \cdots + c_k\, 0 \\ &= 0 \end{aligned}$$

$\square$

Corollary ⊕ A multiple of a solution of a homogeneous linear DE is also a solution.

⊕ A homogeneous linear DE always has the trivial solution $y=0$.

Example: $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solution of
$$x^3 y''' - 2xy' + 4y = 0 \text{ on } (0, \infty).$$
Thus
$$y = c_1 x^2 + c_2 x^2 \ln x$$
is also a solution for any c_1, c_2.

LINEAR INDEPENDENCE.

Defn. A collection of functions $f_1(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that
$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$
for every x in the interval.

If the collection of function is not linearly dependent on the interval, it is said to be **linearly independent**.

Example:

$$f_1(x) = \cos^2 x \quad , f_2(x) = \sin^2 x$$
$$f_3(x) = \sec^2 x \quad , f_4(x) = \tan^2 x$$

are linearly dependent on $(-\pi/2, \pi/2)$ because

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

for $c_1 = c_2 = 1$, $c_3 = -1$, $c_4 = 1$.

Example: $f_1(x) = \sin x$, $f_2(x) = \cos x$ are linearly independent on $(0, \pi)$

Because if

$$c_1 \sin x + c_2 \cos x = 0$$

for any $x \in (0, \pi)$, then

$$c_1 \sin 0 + c_2 \cos 0 = 0$$
$$c_1 \sin(\pi/2) + c_2 \cos(\pi/2) = 0$$

$\Rightarrow c_1 = c_2 = 0$.

SOLUTIONS OF DIFFERENTIAL EQUATIONS

Defn. Suppose each of the functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives.

The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions.

Theorem.

Let $y_1, y_2, \ldots, y_n$ be n solution of the homogeneous linear nth-order differential eq (6) on an interval I. Then the set of solutions is linearly independent on I iff $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x on the interval.

Example:

$$y_1(x) = \frac{\cos(2\ln x)}{x^3}, \quad y_2 = \frac{\sin(2\ln x)}{x^3}$$

are solutions of the eq.

$$x^2 y'' + 7xy' + 13y = 0$$

on $(0, \infty)$.

$$W(y_1(1), y_2(1)) = \begin{vmatrix} 1 & 0 \\ -3 & 2 \end{vmatrix} = 2 \neq 0$$

$\Rightarrow W(y_1(x), y_2(x)) \neq 0 \rightarrow y_1$ and y_2 are linearly independent on $(0, \infty)$.

Defn. Any set $y_1, \ldots, y_n$ of n linearly independent solutions of eq (6) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: (Existence of FSS)

There exists a fundamental set of solutions for the homogeneous linear nth order DE (6) on an interval I.

Theorem. (General solution)

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the DE (6) on an interval I. Then the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, \ldots, c_n$ are arbitrary numbers.

Proof. We prove for $n=2$ (the general case can be treated in an analogous manner).

Let Y be any solution of $a_2 y'' + a_1 y' + a_0 y = 0$. Let y_1 and y_2 be linearly independent of the solution (on the same interval I).

Suppose $x = t \in I$ s.t. $W(y_1(t), y_2(t)) \neq 0$

Assume that

$$Y(t) = k_1 \quad \text{and} \quad Y'(t) = k_2.$$

Since

$$W(y_1(t), y_2(t)) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2(t) \end{vmatrix} \neq 0$$

there exist c_1 and c_2 s.t.

$$c_1 y_1(t) + c_2 y_2(t) = k_1$$

$$c_1 y_1'(t) + c_2 y_2'(t) = k_2$$

Define $G(x) = c_1 y_1(x) + c_2 y_2(x)$, $G(x)$ is a solution of the eq on I. Moreover, $G(x)$ satisfy the initial conditions

$$G(t) = c_1 y_1(t) + c_2 y_2(t) = k_1$$

$$G'(t) = c_1 y_1'(t) + c_2 y_2'(t) = k_2.$$

We have $y(x)$ satisfies the same IVP. However this IVP has a unique solution.

$$\Rightarrow \quad G(x) = y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$\square$

<u>Example:</u> $y_1 = e^{3x}$, $y_2 = e^{-3x}$ are solutions of

$$y'' - 9y = 0 \quad \text{on } (-\infty, \infty).$$

These solutions are linearly independent

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0.$$

Example: $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ are solutions of

$$y''' - 6y'' + 11y' - 6y = 0 \text{ on } (-\infty, \infty)$$

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

$\Rightarrow$ $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ is the general sol.

3. Nonhomogeneous Equations

Theorem. Let y_p be any solution of the non-homogeneous equation:

$$a_n(x) D^n y + \cdots + a_1(x) Dy + a_0(x) y = g(x) \quad (\star)$$

(where $g(x) \neq 0$) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous eq (6) on I. Then the general solution of the eq $(\star)$ is

$$y = c_1 y_1(x) + \cdots + c_n y_n(x) + y_p(x).$$

Proof:

Let $Y(x)$ be any solution of $(\star)$, and a given solution $y_p(x)$ of $(\star)$. Then $Y(x) - y_p(x)$ is a solution of (6).

$\Rightarrow$ $Y(x) - y_p(x)$ is a linear combination of $y_1, \ldots, y_n$. D

The linear combination of $y_1, \dots, y_n$ is called the **complementary function** for $(\star)$.

$$y = \text{complementary function} + \text{any particular sol}$$
$$= y_c + y_p.$$

Example: It is easy to verify that

$$y_p = -\frac{11}{2} - \frac{1}{2}x$$

is a solution of $\quad y''' - 6y'' + 11y' - 6y = 3x$.

$\rightarrow$ the general solution is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x .$$

Theorem. (Superposition Principle)

Let $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ be k particular solutions of the non-homogeneous eq $(\star)$ on an interval I corresponding, in turn, to k distinct functions $g_1, \dots, g_k$. That is y_{p_i} is a particular sol of

$$a_n(x)\, y^{(n)} + \dots + a_0(x)\, y = g_i(x).$$

Then

$$y_p(x) = y_{p_1}(x) + \dots + y_{p_k}(x) \text{ is}$$

a particular solution of

$$a_n(x)\, y^{(n)} + \dots + a_0(x)\, y = g_1(x) + \dots + g_k(x).$$

Example:

$y_{p_1} = -4x^2$ is a sol of $y'' - 3y' + 4y = -16x^2 + 24x - 8$.

$y_{p_2} = e^{2x}$ is a sol of $y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3} = xe^{x}$ is a sol of $y'' - 3y' + 4y = 2xe^{x} - e^{x}$

Then $y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x}$
is a solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^{x} - e^{x}.$$

$\square$